06/10/11 - Lefschetz fibrations and vanishing cycles

Lefschetz fibration = $\pi : (E, \omega) \to B$
symplectic   Riem. surface (for us: D^2 or $\mathbb{C}$)

s.t. $\begin{cases} \bullet \text{ submersion outside of isolated crit pts, with std local model} \\ \bullet \text{ fibers are sympl. submflds.} \end{cases}$

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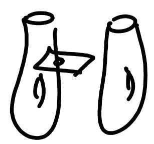
1. Lefschetz fibrations:

Consider: • exact (Liouville) symplectic mflds with corners
(so fibers & B can have boundary).

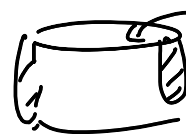
• Equip w/ acs. st. $(E, J) \xrightarrow{\pi} (B, j)$ is (J, j)-holomorphic
(J is not generic ! but $J|_{\text{fibers}}$ can be). (Need convexity : J-hol. curves can't escape (max principle).
$\Rightarrow$ automatically, fibers of π are sympl. submflds

• Symplectic connection: for $x \notin \text{crit}\,\pi$, $TE_x = TE_x^v \oplus TE_x^h$

$TE_x^v = \ker d\pi$ (= tgt to fiber), $TE_x^h = (TE_x^v)^{\perp\omega}$



• Note: $\partial E = \partial^v E \cup \partial^h E$
$\partial^v E = \pi^{-1}(\partial B)$

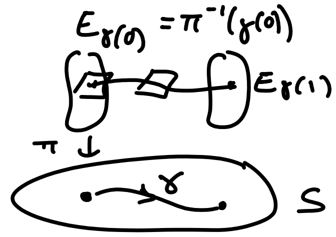


$\partial^h E$ = boundaries of fibers
$\leftarrow \partial^v E$

• Lefschetz fibration: $\pi : (E, \omega_E, J) \to (S, \omega_S, j)$ Riemann surface w/ boundary
s.t. • (J, j)-holomorphic ($\Rightarrow$ fibers symplectic)
• π is a submersion outside of a finite set of isolated, nondegenerate critical pts
where $\exists$ local holom. coords with $\pi : (z_1, \dots z_{n+1}) \mapsto \sum z_i^2$
• We'll always assume E, S are exact symplectic

• Technical conditions: • critical pts are in the interior of E !
• $\partial^h E$ is horizontal ie. if $x \in \partial^h E$, $TE_x^h \subset T_x(\partial^h E)$.
↳ point: can do parallel transport wrt TE_h safely!
• for simplicity, assume critical values are distinct.

- Parallel transport:



$\gamma : [0,1] \hookrightarrow S - \mathrm{critv}(\pi)$

$\Rightarrow P_\gamma : E_{\gamma(0)} \xrightarrow{\sim} E_{\gamma(1)}$ symplectomorphism (exact) i.e. $P^*\theta = \theta$

because:

horiz. sides $\Rightarrow \int_{\square} d\omega = 0 = \int_{\text{left}} \omega - \int_{\text{right}} \omega$.

same for θ...

Ex: (& local model) $Q : \mathbb{C}^{n+1} \to \mathbb{C}$ (ω_{std}, J_{std})

$(z_1, \dots z_{n+1}) \mapsto \sum z_i^2$

need to truncate to get something w/ boundary

Namely $E = \{ z \in \mathbb{C}^{n+1} / \ |Q(z)| \le r, \ k(z) \le s \}$

$\downarrow Q$ $\qquad k(z) = \frac{1}{4}\left(|z|^4 - |\sum z_i^2|^2 \right)$

$D^2(r)$

Check: $TE_z^h = \mathbb{C} \cdot (\bar{z}_1, \dots, \bar{z}_{n+1})$ $\quad \left[\omega(\bar{z}, v) = \frac{1}{2} \sum z_i v_i + \bar{z}_i \bar{v}_i = \frac{1}{2} d(\mathrm{Re}\, Q)(v) \right.$

$\left. \omega(i\bar{z}, v) = \frac{1}{2} d(\mathrm{Im}\, Q)(v) \right]$.

and indeed $dk(\bar{z}) = 0$

$dk(i\bar{z}) = 0$

so this is horizontal ✓

Fact: level sets of Q are $\cong T^*S^n$ or after truncation, $D_sT^*S^n = \{ (x,v) \mid \|v\| \le s \}$.

Namely: $T^*S^n = \{ (v,x) \in \mathbb{R}^{n+1} \times S^n / \langle x, v \rangle = 0 \}$, $\omega = dv \wedge dx$

$\uparrow \simeq$

$Q^{-1}(c)$ $\qquad z \longmapsto \left(-|\mathrm{Re}\, z| \,\mathrm{Im}\, z, \ \frac{\mathrm{Re}(z)}{|\mathrm{Re}(z)|} \right)$

$c \in \mathbb{R}_+$

$\left(\begin{array}{l} \text{observe: } \mathrm{Re}\, Q(z) = |\mathrm{Re}\, z|^2 - |\mathrm{Im}\, z|^2 \\ \mathrm{Im}\, Q(z) = \langle \mathrm{Re}\, z, \mathrm{Im}\, z \rangle \end{array} \right)$

(& these identifications compat. w/ parallel transport along $\mathbb{R}_+$

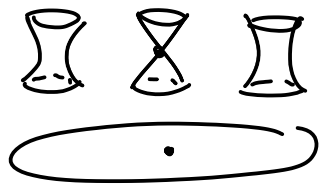
since $d(-|\mathrm{Re}\, z| \,\mathrm{Im}\, z)(v) = -\frac{\langle \mathrm{Re}\, v, \mathrm{Re}\, z \rangle}{|\mathrm{Re}\, z|} \mathrm{Im}\, z - |\mathrm{Re}\, z| \,\mathrm{Im}\, v$

apply to $v = \bar{z}$.

- however $Q^{-1}(0)$ singular at origin

namely identification fails along zero section, i.e. $S^n \subset T^*S^n$ collapsed to critical point

Ex: • $n=1$: $\mathbb{C}^2 \downarrow \mathbb{C}$



fibers: $z_1^2+z_2^2=c \iff uv=c$

$u=z_1+iz_2$
$v=z_1-iz_2$

ie. $\mathbb{C}^*$ or $\mathbb{C}\cup_0\mathbb{C}$.

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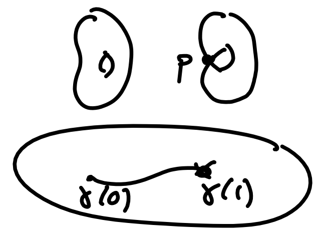
2. Vanishing cycles:



• Vanishing path $=\gamma:[0,1]\to S$
$\gamma(1)\in \mathrm{critval}(\pi)$
γ otherwise disjoint from crit. vals

• v. cycle: $V_\gamma \subset E_{\gamma(0)}$: set of pts s.t. parallel transport along γ converges to crit pt. $p\in E_{\gamma(1)}$

V_γ is an (exact) Lagrangian sphere in $E_{\gamma(0)}$

(Lagr. since collapses under // transport $\Rightarrow \omega_{|V_\gamma}=\omega_{|pt}=0$)

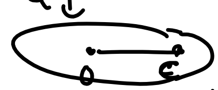
sphere: by local model)

• Lefschetz thimble: $\Delta_\gamma \subset E$: union of parallel transport images of V_γ along γ

smooth Lagr. B^{n+1} in E, $\partial\Delta_\gamma = V_\gamma$.

(smoothness at p not obvious! follows from local model)

Local model. $\mathbb{C}^{n+1} \xrightarrow{Q} \mathbb{C}$



$\gamma=[0,c]\subset\mathbb{R}_+$

$\Rightarrow V_\gamma \simeq$ zero section in T^*S^n

$= \{\mathrm{Im}\, z=0,\ Q(z)=c\}$

$= \{(x_1\dots x_{n+1})\in\mathbb{R}^{n+1} / \sum x_i^2=c\}$

and $\Delta_\gamma = \{(x_1\dots x_{n+1})\in\mathbb{R}^{n+1} / |x|^2\le c\}$.

Even if $\gamma(0)=c\notin\mathbb{R}_+$: $V_\gamma=\sqrt{c}\,S^n$, and $\Delta_\gamma=\bigcup_t \sqrt{\gamma(t)}\,S^n$.